Equivalence of a one-dimensional driven-diffusive system and an equilibrium two-dimensional walk model

Farhad H. Jafarpour* and Somayeh Zeraati

Physics Department, Bu-Ali Sina University, 65174-4161 Hamedan, Iran

It is known that a single product shock measure in some of the one-dimensional driven-diffusive systems with nearest-neighbor interactions might evolve in time quite similarly to a random walker moving on a one-dimensional lattice with reflecting boundaries. The nonequilibrium steady state of the system in this case can be written in terms of a linear superposition of such uncorrelated shocks. Equivalently, one can write the steady state of this system using a matrix-product approach with two-dimensional matrices. In this paper, we present an equilibrium two-dimensional one-transit walk model and find its partition function using a transfer matrix method. We will show that there is a direct connection between the partition functions of these two systems. We will explicitly show that in the steady state, the transfer matrix of the one-transit walk model is related to the matrix representation of the algebra of the driven-diffusive model through a similarity transformation. The physical quantities are also related through the same transformation.

I. INTRODUCTION

During the last couple of years, the time evolution of a single product shock measure in one-dimensional driven-diffusive systems [1–3] was studied in a series of papers [4–9]. It has been observed that by applying some constraints on the microscopic reaction rates of the system, the shock front might have simple random walk dynamics while reflecting from the boundaries of the lattice. In this case, the steady state of the system can be easily obtained by considering a linear superposition of such product measures. Three families of two-state systems with nearest-neighbor interactions was introduced and studied in [5]. This was also generalized in the multispecies systems with long range interactions and even in the systems with discrete time updating scheme [10].

The study of the steady-state properties of one-dimensional driven-diffusive systems has a long history. Different approaches have been used in order to find the stationary-state probability distribution function of these systems ranging from the Bethe ansatz to the matrix-product approach. In the latter scenario, one writes the stationary state of the system as a matrix element of product of noncommuting operators associated with different states of each lattice site of the system. The algebraic relation between these operators, sometimes called the algebra of the system, might have finite- or infinite-dimensional matrix representations [11]. In an attempt to understand the nature of the matrix representations of these algebras, it has been observed that for the systems in which a single product shock measure has a simple random walk dynamics under the time evolution generated by the so-called Hamiltonian of the system, the stationary-state probability distribution function can be expressed by a two-dimensional matrix representation. These matrices contain all of the information about the hopping rates of the shock front and also the densities of the particles

on the left and the right hand side of the shock position. It has been confirmed that the conditions under which the product shock measure has a simple random walk dynamics are exactly those for the existence of a two-dimensional matrix representation for the algebra of the system [12].

Following the paper by Arndt [13], investigations showed that some of the concepts used in the equilibrium statistical mechanics can be extended to the out-of-equilibrium systems. These concepts include the Yang-Lee description of phase transitions in equilibrium systems. It is known that by defining an *ad hoc* partition function in terms of the microscopic reaction rates of an out-of-equilibrium system, one can apply the Yang-Lee approach to spot the transition points. On the other hand, in [14–16], the connection between the steady state of a one-dimensional driven-diffusive system, i.e., the Asymmetric Simple Exclusion Process (ASEP) in random sequential updating scheme with that of a one-transit walk model, was investigated. The investigations were also extended to the case of ASEP with parallel updating scheme in [17].

The authors in [15] showed that by introducing two fugacities associated with the densities of contact points in the equilibrium one-transit walk model, one can clearly explain the phases in the walk model. On the other hand, they showed that if one replaces the two fugacities with the boundary rates of ASEP, a better understanding of the phase diagram of this nonequilibrium system can be obtained. In the present paper, we extend the above mentioned ideas to the case where the shocks with simple random walk dynamics appear in the most general one-dimensional driven-diffusive system. We define a one-dimensional driven-diffusive lattice model with open boundaries in which it is assumed that a single product shock measure can move on the lattice with random walk dynamics. One can introduce an *ad hoc* partition function for this model in terms of its microscopic reaction rates. We also present an equilibrium one-transit walk model. As in [15], the partition function of this model can be calculated by defining proper fugacities. We aim to show that the physical quantities in both models are
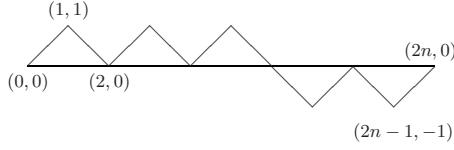
*farhad@ipm.ir

FIG. 1. A typical one-transit walk.

closely related. We believe that the connection between these two systems in the steady state helps us understand the nature of applicability of equilibrium concepts to the nonequilibrium systems.

This paper is organized as follows: we start with the definition of the walk model and calculate some of its relevant physical quantities in terms of two proper fugacities. In order to find the partition function of this model, we will use the transfer matrix method. Then, we consider a general one-dimensional driven-diffusive system and show how its steady-state probability distribution function can be calculated using a matrix-product method and a two-dimensional matrix representation. We finally conclude by showing that the physical quantities of these two systems are closely related if one relates the hopping rates of the shock front in the driven-diffusive system with the fugacities in the walk model.

II. WALK MODEL

Consider a random walker that moves on a two-dimensional path. Each path is defined on a diagonally rotated square lattice. The paths start at $(0,0)$ and end at $(2n,0)$. The walker can only move in the northeast (NE) or in the southeast (SE) direction. The walker can never take two consecutive steps to NE while it might take two consequent steps to SE only once. Each path either crosses the horizontal line only once (in this case, the path can be factorized into two Dyck paths) or never crosses it (in this case, the path is a single Dyck path). According to our definition, there are only $n+1$ different paths of this type. A typical path is illustrated in Fig. 1. As can be seen, the height of each step can be $+1$, 0 or -1. A fugacity z_1 is given to each down step and z_2 to each up step, except those ending on the x axis. The partition function of this model, which contains $2n$ steps, can be easily written as

$$Z_n(z_1,z_2) = (z_1 z_2)^n \widetilde{Z}_n(z_1,z_2), \quad (1)$$

in which

$$\widetilde{Z}_n(z_1,z_2) = \sum_{q=0}^{n} z_1^{-q} z_2^{-n+q} = \frac{z_1^{-n} z_2 - z_2^{-n} z_1}{z_2 - z_1}. \quad (2)$$

The expression (2) has a simple interpretation given in [16]. The weight $z_1^{-q} z_2^{-n+q}$, associated with a path containing q contacts with the x axis from above, is obtained by assigning a factor z_1^{-1} to each contact with the x axis from above, excluding an initial upward step, and a factor z_2^{-1} to each contact from below, excluding a final upward step. As explained in [16], the prefactor $(z_1 z_2)^n$ does not change the critical behavior of the walk model.
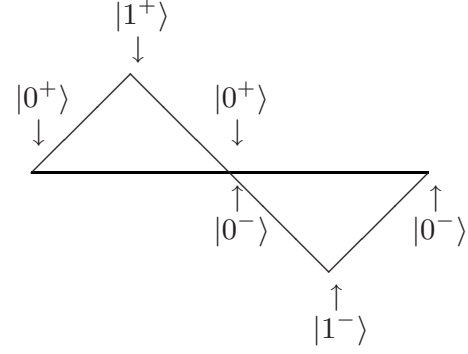

FIG. 2. Assigning the base vectors to the vertices.

In what follows, we will show that the partition function $\widetilde{Z}_n(z_1,z_2)$ can be written as

$$\widetilde{Z}_n(z_1,z_2) = \langle L|T^n|R\rangle, \quad (3)$$

in which T is a two-step transfer matrix. We associate T° to each odd step and T^e to each even step and write $T=T^{\circ}T^e$. In order to find the transfer matrix T and the vectors $|R\rangle$ and $\langle L|$, we first introduce four base vectors $|s_1\rangle=|1^+\rangle$, $|s_2\rangle=|1^-\rangle$, $|s_1'\rangle=|0^+\rangle$, and $|s_2'\rangle=|0^-\rangle$ with the following properties:

$$\sum_{\{s_i\}} |s_i\rangle\langle s_i| = \sum_{\{s_i'\}} |s_i'\rangle\langle s_i'| = \mathcal{I}, \quad (4)$$

in which $\mathcal{I}$ is an identity 2×2 matrix. Using the completeness of the base vectors, the partition function $\widetilde{Z}_n(z_1,z_2)$ can be expanded as follows:

$$\widetilde{Z}_n(z_1,z_2) = \sum_{\{s_i',s_j',\cdots,s_{n+1}',s_i,s_j,\cdots,s_n\}} \langle L|s_i'\rangle\langle s_i'|T^{\circ}|s_i\rangle\langle s_i|T^e|s_j'\rangle$$
$$\times \langle s_j'|T^{\circ}|s_j\rangle\langle s_j|\cdots|s_{n+1}'\rangle\langle s_{n+1}'|R\rangle. \quad (5)$$

The matrix representations for the base vectors can be chosen as

$$|1^+\rangle = |0^+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1^-\rangle = |0^-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (6)$$

We assign these base vectors to the different vertices of each path according to the Fig. 2. Now, according to the definition of the model, only the following transitions are allowed:

$$\langle 1^+|T^e|0^+\rangle = \frac{1}{z_1}, \quad \langle 1^+|T^e|0^-\rangle = 0,$$

$$\langle 1^-|T^e|0^+\rangle = 0, \quad \langle 1^-|T^e|0^-\rangle = \frac{1}{z_2},$$

$$\langle 0^+|T^{\circ}|1^+\rangle = 1, \quad \langle 0^+|T^{\circ}|1^-\rangle = 1,$$

$$\langle 0^-|T^{\circ}|1^+\rangle = 0, \quad \langle 0^-|T^{\circ}|1^-\rangle = 1,$$

$$\langle 0^+|R\rangle = 1, \quad \langle 0^-|R\rangle = 1,$$

$$\langle L | 0^+ \rangle = 1, \quad \langle L | 0^- \rangle = 0. \tag{7}$$

These equations determine the transfer matrices T^o and T^e

$$T^o = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T^e = \begin{pmatrix} \dfrac{1}{z_1} & 0 \\ 0 & \dfrac{1}{z_2} \end{pmatrix}, \tag{8}$$

and also the vectors $\langle L |$ and $| R \rangle$

$$\langle L | = (1,0), \quad | R \rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{9}$$

Now, using Eqs. (8) and (9), one can readily see that Eq. (3) gives Eq. (2).

Following [15], we define two contact operators $\hat{a}_i = |1^+\rangle_{2i-1}\langle 0^+|_{2i}$ and $\hat{b}_i = |1^-\rangle_{2i-1}\langle 0^-|_{2i}$ associated with a contact from above and below the x axis, respectively. Now, one can calculate the probability of finding a contact at site $2i$ above or below the horizontal line as follows:

$$\langle \hat{a}_i \rangle_n = \frac{\langle L | T^i \hat{a}_i T^{n-i} | R \rangle}{\langle L | T^n | R \rangle}, \tag{10}$$

$$\langle \hat{b}_i \rangle_n = \frac{\langle L | T^i \hat{b}_i T^{n-i} | R \rangle}{\langle L | T^n | R \rangle}. \tag{11}$$

It is now easy to check that these probabilities can be written as

$$\langle \hat{a}_i \rangle_n = \frac{\widetilde{Z}_i(z_1, \infty)\widetilde{Z}_{n-i}(z_1, z_2)}{\widetilde{Z}_n(z_1, z_2)}, \tag{12}$$

$$\langle \hat{b}_i \rangle_n = \frac{\widetilde{Z}_{i-1}(z_1, z_2)\widetilde{Z}_{n-i+1}(\infty, z_2)}{\widetilde{Z}_n(z_1, z_2)}. \tag{13}$$

Using the fact that $\widetilde{Z}_n(z_1, z_2) = \widetilde{Z}_n(z_2, z_1)$, it can be seen that these probabilities have the following symmetry:

$$\langle \hat{a}_i \rangle_n(z_1, z_2) = \langle \hat{b}_{n-i+1} \rangle_n(z_2, z_1), \tag{14}$$

which can be understood from the symmetry of the paths. Let us define the average number of contacts from above and below the horizontal axis at position $x = \frac{i}{n}$ ($0 \le x \le 1$) as

$$\langle \hat{a}_x \rangle = \langle \hat{a}_{xn} \rangle_n, \quad \langle \hat{b}_x \rangle = \langle \hat{b}_{xn} \rangle_n. \tag{15}$$

Now, in the thermodynamic limit $n \to \infty$ and in different phases of the one-transit walk model, we find for average number of contacts from above x axis

$$\langle \hat{a}_x \rangle = \begin{cases} 0 & \text{for} \quad z_1 > z_2, \\ 1 & \text{for} \quad z_2 > z_1, \\ 1-x & \text{for} \quad z_1 = z_2 \end{cases} \tag{16}$$

and for the average number of contacts from below x axis

$$\langle \hat{b}_x \rangle = \begin{cases} 1 & \text{for} \quad z_1 > z_2, \\ 0 & \text{for} \quad z_2 > z_1, \\ x & \text{for} \quad z_1 = z_2. \end{cases} \tag{17}$$

In the following sections, we present a general one-dimensional driven-diffusive system with open boundaries. We discuss the case when a single product shock measure can develop in the system. The steady-state probability distribution will be given using a matrix method. Finally, we investigate its connections with the walk model.

III. SHOCK IN A DRIVEN-DIFFUSIVE SYSTEM

Consider a driven-diffusive system of classical particles defined on a one-dimensional lattice of length n with open boundaries. For simplicity, we consider the case where each lattice site can be occupied by a single particle or it can be empty, although it is easy to generalize the idea to the systems with more than one species of particles. The time evolution of the probability distribution function of any configuration of a Markovian interacting particle system $|P(t)\rangle$ is governed by a master equation, which can be written as a Schrödinger-like equation in imaginary time

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle, \tag{18}$$

in which H is called the Hamiltonian [3]. The matrix elements of the Hamiltonian H are the transition rates between different configurations. For a system with nearest-neighbor interactions, the Hamiltonian H has the following general form:

$$H = \sum_{k=1}^{n-1} h_{k,k+1} + h_1 + h_n. \tag{19}$$

in which

$$h_{k,k+1} = \mathcal{I}^{\otimes(k-1)} \otimes h \otimes \mathcal{I}^{\otimes(n-k-1)}, \tag{20}$$

$$h_1 = h^{(l)} \otimes \mathcal{I}^{\otimes(n-1)}, \tag{21}$$

$$h_n = \mathcal{I}^{\otimes(n-1)} \otimes h^{(r)}. \tag{22}$$

For our two-states system, $\mathcal{I}$ is a 2×2 identity matrix and h is a 4×4 matrix describing the bulk interactions. The two matrices $h^{(r)}$ and $h^{(l)}$ are both 2×2 matrices, which describe the interactions at the boundaries.

A shock in such a system is defined as a sharp discontinuity between a high-density and a low-density region. We assume that a single product shock measure in the system defined as

$$|k\rangle = \begin{pmatrix} 1-\rho_1 \\ \rho_1 \end{pmatrix}^{\otimes k} \otimes \begin{pmatrix} 1-\rho_2 \\ \rho_2 \end{pmatrix}^{\otimes n-k}, \tag{23}$$

in which k ($0 \le k \le n$) is the position of the shock, can evolve in time generated by the Hamiltonian of the system. The quantities ρ_1 and ρ_2 are the densities of the particles on the left-hand side and the right-hand side of the shock posi-

tion. In [12], the authors have studied the case where the shock position in Eq. (23) has a simple random walk dynamics in the bulk of the system, while reflecting from the left and the right boundaries, by applying some constraints on the microscopic reaction rates of the system. If the shock front hops to the left (right) with the rate $\delta_l(\delta_r)$, we assume that it reflects from the right (left) boundary with the same rate. As we will discuss, this will not alter the generality of the problem.

The steady state of the system it this case can be written as a linear superposition of the shocks (23). In the same reference, the authors also showed that the same steady state can be obtained using a matrix-product approach. According to this approach, the steady state of the system $|P^*\rangle$ can be written as

$$|P^*\rangle = \frac{1}{Z_n}\langle W| \begin{pmatrix} D \\ E \end{pmatrix}^{\otimes n} |V\rangle, \qquad (24)$$

provided that one chooses the operators E and D, associated with the presence of a hole and a particle at each lattice site, as follows:

$$E = \begin{pmatrix} (1-\rho_1) & d_0 \\ 0 & \frac{\delta_l}{\delta_r}(1-\rho_2) \end{pmatrix}, \quad D = \begin{pmatrix} \rho_1 & -d_0 \\ 0 & \frac{\delta_l}{\delta_r}\rho_2 \end{pmatrix}. \quad (25)$$

The matrix elements of the vectors $|V\rangle$ and $\langle W|$, defined as

$$|V\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \langle W| = (w_1 \quad w_2), \qquad (26)$$

besides d_0 should only satisfy two constraints

$$d_0\frac{w_1}{w_2} = \rho_2 - \rho_1, \quad d_0\frac{v_2}{v_1} = \left(\frac{\delta_l}{\delta_r}\right)(\rho_1 - \rho_2). \qquad (27)$$

The normalization factor in Eq. (24) is given by $Z_n = \langle W|(D+E)^n|V\rangle$. One should note that the only difference between the representation presented here with the one used in [12] is that here, we have chosen the *upper triangular matrices* instead of *lower triangular matrices*.

IV. CONNECTION BETWEEN THE WALK MODEL AND THE DRIVEN-DIFFUSIVE SYSTEM

In what follows, we show that the matrix-product partition function defined in the previous section is exactly equal to the partition function of the walk model $\widetilde{Z}_n(z_1,z_2)$ defined in Eq. (3), provided that the fugacities of the contact point are chosen as

$$z_1 = 1, \quad z_2 = \frac{\delta_r}{\delta_l}. \qquad (28)$$

Defining $C := D+E$ and using a similarity transformation, we can write

$$Z_n = \langle \widetilde{W}|\widetilde{C}^n|\widetilde{V}\rangle,$$

in which

$$\widetilde{C} = U^{-1}CU, \quad |\widetilde{V}\rangle = U^{-1}|V\rangle, \quad \langle \widetilde{W}| = \langle W|U. \qquad (29)$$

We introduce U as

$$U = \begin{pmatrix} u_1 & \dfrac{\delta_l}{\delta_r} \\ & 1 - \dfrac{\delta_l}{\delta_r}u_1 \\ 0 & u_2 \end{pmatrix}, \qquad (30)$$

in which u_1 and u_2 are assumed to be nonzero. Since the matrix elements of the vectors $\langle W|$ and $|V\rangle$ should only satisfy Eq. (26), we define these elements as

$$v_1 = \frac{u_1}{1-\dfrac{\delta_l}{\delta_r}}, \quad v_2 = u_2, \quad w_1 = \frac{1}{u_1}, \quad w_2 = \frac{1}{u_2}\begin{pmatrix} -\dfrac{\delta_l}{\delta_r} \\ 1-\dfrac{\delta_l}{\delta_r} \end{pmatrix}, \qquad (31)$$

provided that

$$d_0\frac{u_2}{u_1} = (\rho_1 - \rho_2)\begin{pmatrix} \dfrac{\delta_l}{\delta_r} \\ 1-\dfrac{\delta_l}{\delta_r} \end{pmatrix}. \qquad (32)$$

Now, it is easy to verify that

$$\widetilde{C} = T, \quad |\widetilde{V}\rangle = |R\rangle, \quad \langle \widetilde{W}| = \langle L|, \qquad (33)$$

which means that the partition function of the walk model $\widetilde{Z}_n(z_1=1, z_2=\frac{\delta_r}{\delta_l})$ is equal to the normalization factor of the driven-diffusive system obtained from the matrix-product approach in Eq. (24).

Finally, we show that the density profile of the particles in the driven-diffusive system in terms of the matrix-product approach defined as

$$\langle \rho_i \rangle = \frac{\langle W|C^{i-1}DC^{n-i}|V\rangle}{\langle W|C^n|V\rangle}, \quad 1 \le i \le n \qquad (34)$$

can be expressed in terms of the probability of finding a contact above or below the x axis defined in Eqs. (10) and (11). Using the same similarity transformation mentioned above, one finds

$$\langle \rho_i \rangle = \frac{\langle L|T^{i-1}\widetilde{D}T^{n-i}|R\rangle}{\langle L|T^n|R\rangle}, \qquad (35)$$

in which $\widetilde{D} = U^{-1}DU$. Now, it is a straightforward step to show that the density profile of the particles can be rewritten as

$$\langle \rho_i \rangle = \rho_1\langle \hat{a}_i \rangle_n + \rho_2\langle \hat{b}_i \rangle_n, \qquad (36)$$

given that Eq. (32) is satisfied. Note that in terms of the fugacities z_1 and z_2, the phase diagram of both systems can be explained as follows: for $z_1 > z_2$ ($z_1 < z_2$) and in the thermodynamic limit $n \to \infty$, the walk has only contacts from

below (above) the horizontal axis. This is equivalent to the fact that the shock front has a tendency to move to the left (right), which in turns means that the mean density of the particles in the bulk of the driven-diffusive system is equal to ρ_2 (ρ_1). One can see that these are confirmed by Eq. (36). On the other hand, on the coexistence line, one has $z_1 = z_2$ or in terms of the shock position, hopping rates $\delta_l = \delta_r$. On this line, the shock position can be anywhere on the lattice with equal probabilities, which results in a linear density profile for the particles on the lattice. Using Eqs. (16) and (17), one can see that Eq. (36) gives the right density profile.

V. CONCLUSION

In this paper, we considered two different systems: a one-transit walk model defined on a diagonally rotated square lattice and a general driven-diffusive system in which a single product shock measure has a simple random walk dynamics. We showed that how the phase diagrams of these two systems are connected by associating the fugacities in the walk model with the hopping rates of the shock front in the driven-diffusive system. In fact, following the idea of [15], one can consider the normalization factor of a nonequilibrium system as a partition function of a two-dimensional equilibrium model. The fugacities defined in the walk model are related to the hopping rate of the shock front. The main difference of this paper with those previous is that our example belongs to a larger family of driven-diffusive systems instead of just the ASEP as an example. The physical quantities of these two systems are directly related [for instance, see Eq. (36)]. As a simple generalization, it can be shown that by redefining the fugacities of the first and the last steps in the walk model, one can also generate the partition function of the driven-diffusive systems in the case where the hopping rates of the shock front in the bulk are different from those of the boundaries. As another generalization to this work, one can try to investigate the driven-diffusive systems in a discrete time updating scheme and find the proper walk models which are related to them.

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